# Spinor algebras

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## Abstract

We consider supersymmetry algebras in space–times with arbitrary signature and minimal number of spinor generators. The interrelation between super Poincaré and super conformal algebras is elucidated. Minimal super conformal algebras are seen to have as bosonic part a classical semisimple algebra naturally associated to the spin group. This algebra, the $\mathrm{Spin}(s, t)$-algebra, depends both on the dimension and on the signature of space–time. We also consider maximal super conformal algebras, which are classified by the orthosymplectic algebras. © 2001 Elsevier Science B.V. All rights reserved.

## 1. Introduction

In recent times the extension of Poincaré and conformal superalgebras to orthosymplectic algebras has been considered with a variety of purposes. In particular the role of $\mathrm{osp}(1|32, \mathbb{R})$ and $\mathrm{osp}(1|64, \mathbb{R})$ as minimal superalgebras containing the conformal algebras in 10 and 11 dimensions (or the anti de Sitter algebra in 11 and 12 dimensions) has been considered in view of possible generalizations of M-theory [1–3,7,8,16–18,20,21] and of string theory to F-theory [9]. The contractions of orthosymplectic algebras are used in the study of BPS branes [7,12,19,34,35].

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In the present paper we address the more general question of whether such extensions are possible for space–times with Lorentz group $SO(s, t)$. Space–times with more than one time direction have been studied in order to unify duality symmetries of string and M theories [10,15] and to explore BPS states in two-times physics [6,10]. A theory based on the gauging of orthosymplectic algebras has been suggested as a non-perturbative definition of M-theory [16].

Supersymmetric extensions of Poincaré and conformal (or anti de Sitter) algebras in higher dimensional spaces have been considered in the literature [3,5,7,20,22,23]. Our analysis embraces all possible dimensions and signatures, so we will make contact with the previous investigations.

We first consider $N = 1$ super Poincaré algebras for arbitrary space–time signature and dimension, extending the usual classification of supersymmetries in any dimension [22]. We then compute the orthosymplectic superalgebras containing $so(s, t)$ as a subalgebra of the symplectic algebra. The embedding we look for is such that the symplectic fundamental representation is an irreducible spinor representation when restricted to the orthogonal algebra. Orthosymplectic superalgebras are seen to contain Poincaré supersymmetry, either as a subalgebra or as a Wigner–Inonü contraction. This generalizes the fact that the M-theory superalgebra can be seen, either as a contraction of $osp(1|32, \mathbb{R})$ or as a subalgebra of $osp(1|64, \mathbb{R})$.

The paper is organized as follows. In Section 2 we review properties of spinors and Clifford algebras for arbitrary signature and dimension and set up the notation for the rest of the paper. We also provide the symmetry properties of the morphisms which allow us the classification of space–time superalgebras. In Sections 3 and 4 Poincaré and conformal supersymmetry are studied in a uniform way. In Section 5 the orthosymplectic algebras and their contractions to centrally extended super Poincaré and super translation algebras are studied. In Section 6 we introduce the concept of orthogonal symplectic and linear spinors which, together with the reality properties allows us to associate a real simple algebra from the classical series to the Spin group (called Spin$(s, t)$-algebra). In Section 7 we show that the minimal super conformal algebras are supersymmetric extensions of the Spin$(V)$-algebra. A maximal superalgebra with the same number of odd generators is always an orthosymplectic algebra. In Section 8 we summarize our results and retrieve the examples of Minkowskian signature.

## 2. Properties of spinors of SO*(V)*

Let $V$ be a real vector space of dimension $D = s + t$ and $\{v_\mu\}$ a basis of it. On $V$ there is a non-degenerate symmetric bilinear form which in the basis is given by the matrix

$$\eta_{\mu\nu} = \mathrm{diag}(+, \ldots (s \text{ times}) \ldots, +, -, \ldots (t \text{ times}) \ldots, -).$$

We consider the group Spin$(V)$, the unique double covering of the connected component of $SO(s, t)$ and its spinor representations. A spinor representation of Spin$(V)^{\mathbb{C}}$ is an irreducible complex representation whose highest weights are the fundamental weights corresponding to the right extreme nodes in the Dynkin diagram. These do not descend to representations of $SO(V)$. A spinor type representation is any irreducible representation

that does not descend to $SO(V)$. A spinor representation of $Spin(V)$ over the reals is an irreducible representation over the reals whose complexification is a direct sum of spin representations.

Two parameters, the signature $\rho \bmod(8)$ and the dimension $D \bmod(8)$ classify the properties of the spinor representation. Through this paper we will use the following notation:

$$\rho = s - t = \rho_0 + 8n, \qquad D = s + t = D_0 + 8p,$$

where $\rho_0, D_0 = 0, \ldots, 7$. We set $m = p - n$, so

$$s = \tfrac{1}{2}(D + \rho) = \tfrac{1}{2}(\rho_0 + D_0) + 8n + 4m, \qquad t = \tfrac{1}{2}(D - \rho) = \tfrac{1}{2}(D_0 - \rho_0) + 4m.$$

The signature $\rho \bmod(8)$ determines if the spinor representations are real ($\mathbb{R}$), quaternionic ($\mathbb{H}$) or complex ($\mathbb{C}$) type.

The dimension $D \bmod(8)$ determines the nature of the $Spin(V)$-morphisms of the spinor representation $S$. Let $g \in Spin(V)$ and let $\Sigma(g) : S \to S$ and $L(g) : V \to V$ be the spinor and vector representations of $l \in Spin(V)$, respectively. Then a map $A$

$$A : S \otimes S \to \Lambda^k,$$

where $\Lambda^k = \Lambda^k(V)$ are the $k$-forms on $V$, is a $Spin(V)$-morphism if

$$A(\Sigma(g)s_1 \otimes \Sigma(g)s_2) = L^k(g)A(s_1 \otimes s_2).$$

In the next subsections we analyze the properties of spinors for arbitraries $\rho$ and $D$.

### 2.1. Spinors and Clifford algebras

We denote by $\mathcal{C}(s, t)$ the Clifford algebra associated to $V$ and $\eta$. It is defined as the real associative algebra generated by the symbols $\mathcal{I}$, $\Gamma_\mu$ with relations

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu}\mathcal{I}, \tag{1}$$

and with $\mathcal{I}$ the unit element.

Let $\mathbb{C}(p)$ be the algebra of $p \times p$ complex matrices. The complexification of the Clifford algebra, $\mathcal{C}(s, t)^{\mathbb{C}} \simeq \mathcal{C}(t, s)^{\mathbb{C}}$, is isomorphic to $\mathbb{C}(2^{D/2})$ for $D$ even and to $\mathbb{C}(2^{(D-1)/2}) \oplus \mathbb{C}(2^{(D-1)/2})$ for $D$ odd. The real Clifford algebras are isomorphic to certain matrix algebras. They are classified by $\rho = s - t \bmod(8)$ (see [13,14,24–27]). Notice that $D$ and $\rho$ have always the same parity. We list the results in Table 1, where we have used the following notation: $\mathbf{2} \times E = E \oplus E$, $\mathbb{R}(p)$ and $\mathbb{C}(p)$ mean the algebra of $p \times p$ matrices with entries in the real or complex numbers, respectively. $\mathbb{H}(p)$ instead means the set of $p \times p$ complex

Table 1
Clifford algebras

| | $\rho$ even | | | | $\rho$ odd | | | |
|---|---|---|---|---|---|---|---|---|
| | 0 | 2 | 4 | 6 | 1 | 3 | 5 | 7 |
| $\mathcal{C}(s, t)$ | $\mathbb{R}(2^{D/2})$ | $\mathbb{R}(2^{D/2})$ | $\mathbb{H}(2^{D/2-1})$ | $\mathbb{H}(2^{D/2-1})$ | $\mathbf{2}\mathbb{R}(2^{(D-1)/2})$ | $\mathbb{C}(2^{(D-1)/2})$ | $\mathbf{2}\mathbb{H}(2^{(D-1)/2})$ | $\mathbb{C}(2^{(D-1)/2})$ |

matrices satisfying the quaternionic condition,

$$M^* = -\Omega M \Omega, \tag{2}$$

where $\Omega$ is the symplectic metric. This means that $p$ is even and that $M$ can be written as a $\frac{1}{2}p \times \frac{1}{2}p$ matrix whose entries are quaternionic. Using the two-dimensional complex representation of the quaternions we recover the previous description. We stress that all the algebras appearing in Table 1 are taken as real algebras. The real dimension of the Clifford algebra is $2^D$ in all cases.

We consider a representation of the Clifford algebra in a vector space $S$ of dimension $2^{D/2}$ for $D$ even and $2^{(D-1)/2}$ for $D$ odd, as given by Table 1. This representation is faithful except for $\rho = 1, 5 \bmod(8)$. We will denote by $\gamma_\mu$ the images of the generators $\Gamma_\mu$ by this representation. From Table 1 one can see also when these matrices are real, quaternionic or just complex. $S$ is then a real, quaternionic or complex vector space.

It is clear that in general $\mathcal{C}(s, t)$ and $\mathcal{C}(t, s)$ are not isomorphic. However, the Clifford algebras have a natural $\mathbb{Z}_2$ grading, being the degree of $\Gamma_\mu$ equal to 1. The relations (1) are homogeneous in this degree. The even (degree zero) part $\mathcal{C}^+(s, t)$ is a subalgebra generated by products of an even number of elements of the basis $\Gamma_\mu$. It is then true that $\mathcal{C}^+(s, t) \simeq \mathcal{C}^+(t, s)$. The Lorentz generators are products of two elements, so it follows trivially that $\mathrm{so}(s, t) \simeq \mathrm{so}(t, s)$. This will be important since we are in fact interested in the irreducible representations of $\mathrm{Spin}(V)$.

For $D$ odd the representation $S$ of the Clifford algebra is irreducible under $\mathrm{Spin}(V)$. It is a spinor representation. For $D$ even, it splits into two irreducible spinor representations (called Weyl or chiral spinors) $S = S^+ \oplus S^-$ of half the dimension.

We consider first the odd cases. Since for our purposes only $|\rho_0|$ is important, we will have up to two possible Clifford algebras in each case.

$|\rho_0| = 1$. The Clifford algebras are the ones of $\rho_0 = 1, 7$. We see that $\rho = 1$ gives directly a real representation of real dimension $2^{(D-1)/2}$.

$|\rho_0| = 3$. The two possibilities are $\rho_0 = 3, 5$. $\rho_0 = 5$ gives a quaternionic representation of complex dimension $2^{(D-1)/2}$.

$|\rho_0| = 5$. As the case $|\rho_0| = 3$.

$|\rho_0| = 7$. As the case $|\rho_0| = 1$.

We consider now the even cases.

$|\rho_0| = 0$. There is only one possibility, $\rho_0 = 0$. The representation is real of dimension $2^{D/2}$. The projections on $S^\pm$ are also real. This is because the projectors are

$$P^\pm = \tfrac{1}{2}(1 \pm \gamma_{D+1}),$$

where $\gamma_{D+1} = \gamma_1 \cdots \gamma_D$, which is also real.

$|\rho_0| = 2$. The two possibilities are $\rho_0 = 2, 6$. $\rho = 2$ has a real representation, and $\rho = 6$ has a quaternionic representation. But the projectors in each case are neither real nor quaternionic,

$$P^\pm = \tfrac{1}{2}(1 \pm \mathrm{i}\gamma_{D+1}),$$

so the representations $S^\pm$ are just complex.

Table 2
Properties of spinors

| $\rho_0$ (odd) | Real dim$(S)$ | Reality | $\rho_0$ (even) | Real dim$(S^\pm)$ | Reality |
|---|---|---|---|---|---|
| 1 | $2^{(D-1)/2}$ | $\mathbb{R}$ | 0 | $2^{D/2-1}$ | $\mathbb{R}$ |
| 3 | $2^{(D+1)/2}$ | $\mathbb{H}$ | 2 | $2^{D/2}$ | $\mathbb{C}$ |
| 5 | $2^{(D+1)/2}$ | $\mathbb{H}$ | 4 | $2^{D/2}$ | $\mathbb{H}$ |
| 7 | $2^{(D-1)/2}$ | $\mathbb{R}$ | 6 | $2^{D/2}$ | $\mathbb{C}$ |

$|\rho_0| = 4$. There is only one possibility, $\rho_0 = 4$. The representation is quaternionic of complex dimension $2^{D/2}$. The projectors are

$$P^\pm = \tfrac{1}{2}(1 \pm \gamma_{D+1}),$$

which is quaternionic, so $S^\pm$ are also quaternionic representations.

$|\rho_0| = 6$. As the case $|\rho| = 2$.

In Table 2 we summarize all these properties together with the real dimension of the spinor representation.

Space–time supersymmetry algebras are real superalgebras. The odd generators are in spinor representations of the Lorentz group, so we need to use real spinor representations. For each case, real quaternionic or complex, we use an irreducible real spinor representation, with the dimension indicated in Table 2.

*Real case, $\rho_0 = 0, 1, 7$.* Let $S$ be a finite dimensional complex vector space. A conjugation $\sigma$ is a $\mathbb{C}$-antilinear map $\sigma : S \to S$,

$$\sigma(as_1 + bs_2) = a^*\sigma(s_1) + b^*\sigma(s_2), \quad a, b \in \mathbb{C}, \quad s_i \in S,$$

such that $\sigma^2 = \mathcal{I}$. Let $S$ be the vector space of an irreducible spinor representation of Spin$(V)$. In this case there is a conjugation $\sigma$ that commutes with Spin$(V)$,

$$\sigma(gs) = g\sigma(s), \quad g \in \text{Spin}(V),$$

and then Spin$(V)$ acts on the real vector space $S^\sigma = \{s \in S | \sigma(s) = s\}$. The spinor representation is an irreducible representation of type $\mathbb{R}$.

*Quaternionic case, $\rho_0 = 3, 4, 5$.* A pseudoconjugation is an antilinear map on $S$ such that $\sigma^2 = -\mathcal{I}$. $S$ has necessarily even dimension. If we have a real Lie algebra with an irreducible representation, one can prove that it is of quaternionic type if and only if there exists a pseudoconjugation commuting with the action of the Lie algebra. So a quaternionic representation of Spin$(V)$ has a pseudoconjugation $\sigma$. The condition $\sigma(gs) = g\sigma(s)$ is equivalent, in a certain basis of $S \simeq \mathbb{C}^{2n}$, to (2).

Let $S$ be a quaternionic representation of Spin$(V)$. We take $\tilde{S} \simeq S \oplus S \simeq S \otimes W$, with $W = \mathbb{C}^2$. On $S \otimes W$ we can define a conjugation $\tilde{\sigma} = \sigma \otimes \sigma_0$, with $\sigma_0$ a pseudoconjugation on $W$. In a basis of $W$ we can choose $\sigma_0(w) = \Omega w^*$, with

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The group $\mathrm{SU}(2) = \mathrm{USp}(2)$ commutes with $\sigma_0$, so we have that $\mathrm{Spin}(V) \otimes \mathrm{SU}(2)$ commutes with $\tilde{\sigma}$ and has a real representation on

$$\tilde{S}^\sigma = \{t \in \tilde{S} | \tilde{\sigma}(t) = t\}.$$

We note at this point that there is a smaller group, $\mathrm{SO}^*(2) \simeq \mathrm{SO}(2) \simeq \mathrm{U}(1)$ contained in $\mathrm{SU}(2)$. It will play a role in the construction of superalgebras.

*Complex case, $\rho_0 = 2, 6$.* The representation of the Clifford algebra $\mathcal{C}(s, t)$ on $S = S^+ \oplus S^-$ for $\rho_0 = 2$ is real. This means that it has a conjugation which commutes with the action of $\mathcal{C}(s, t)$. For $\rho_0 = 6$ the Clifford algebra is quaternionic, which means that it has a pseudoconjugation. Nevertheless, the orthogonal group $\mathrm{Spin}(s, t)$ is isomorphic to $\mathrm{Spin}(t, s)$, so we can use the Clifford algebra $\mathcal{C}(t, s)$ which has $\rho_0 = 2$ and a conjugation.

We conclude then that for $\rho_0 = 2, 6$ there is a conjugation $\sigma$ on $S$ commuting with the action of $\mathrm{Spin}(V)$. If follows that there is a representation of $\mathrm{Spin}(V)$ on the real vector space $S^\sigma$.

In particular, we have that $\sigma(S^\pm) = S^\mp$. We can define an action of $\mathrm{U}(1)$ on $S$,

$$\mathrm{e}^{\mathrm{i}\alpha}(s^+ \oplus s^-) = \mathrm{e}^{\mathrm{i}\alpha} s^+ \oplus \mathrm{e}^{-\mathrm{i}\alpha} s^-.$$

This action commutes also with $\sigma$, so it is defined on $S^\sigma$.

The groups $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ appearing in the quaternionic and complex case, respectively, are referred to as R-symmetry groups.

### 2.2. $\mathrm{Spin}(V)$-morphisms

The symmetry properties of the $\mathrm{Spin}(V)$-morphisms

$$S \otimes S \to \Lambda^k$$

depend on $D \bmod(8)$, and are listed in Table 3. We put $-1$ if the morphism is antisymmetric, $+1$ if it is symmetric and leave it blank if no symmetry properties can be defined. Notice that one can restrict $k$ to $2k + 1 \leq D$ since $\Lambda^k \simeq \Lambda^{(D-k)}$ are isomorphic as $\mathrm{Spin}(V)$-modules. This table can be obtained exactly as Table 1.5.1 in [28], using the formalism of [28].

Table 3
Properties of morphisms

| $D$ | $k$ even | | $k$ odd | |
|---|---|---|---|---|
| | Morphism | Symmetry | Morphism | Symmetry |
| 0 | $S^\pm \otimes S^\pm \to \Lambda^k$ | $(-1)^{k(k-1)/2}$ | $S^\pm \otimes S^\mp \to \Lambda^k$ | |
| 1 | $S \otimes S \to \Lambda^k$ | $(-1)^{k(k-1)/2}$ | $S \otimes S \to \Lambda^k$ | $(-1)^{k(k-1)/2}$ |
| 2 | $S^\pm \otimes S^\mp \to \Lambda^k$ | | $S^\pm \otimes S^\pm \to \Lambda^k$ | $(-1)^{k(k-1)/2}$ |
| 3 | $S \otimes S \to \Lambda^k$ | $-(-1)^{k(k-1)/2}$ | $S \otimes S \to \Lambda^k$ | $(-1)^{k(k-1)/2}$ |
| 4 | $S^\pm \otimes S^\pm \to \Lambda^k$ | $-(-1)^{k(k-1)/2}$ | $S^\pm \otimes S^\mp \to \Lambda^k$ | |
| 5 | $S \otimes S \to \Lambda^k$ | $-(-1)^{k(k-1)/2}$ | $S \otimes S \to \Lambda^k$ | $-(-1)^{k(k-1)/2}$ |
| 6 | $S^\pm \otimes S^\mp \to \Lambda^k$ | | $S^\pm \otimes S^\pm \to \Lambda^k$ | $-(-1)^{k(k-1)/2}$ |
| 7 | $S \otimes S \to \Lambda^k$ | $(-1)^{k(k-1)/2}$ | $S \otimes S \to \Lambda^k$ | $-(-1)^{k(k-1)/2}$ |

Let $S^\vee$ be the dual space of $S$ and let $C_\pm : S \to S^\vee$ be the map intertwining two equivalent representations of the Clifford algebra, namely

$$C_+^{-1}\gamma_\mu C_+ = \gamma_\mu^{\mathrm{T}} \quad \text{for } D = 1 \bmod(4), \qquad C_-^{-1}\gamma_\mu C_- = -\gamma_\mu^{\mathrm{T}} \quad \text{for } D = 3 \bmod(4),$$
$$C_\pm^{-1}\gamma_\mu C_\pm = \pm\gamma_\mu^{\mathrm{T}} \quad \text{for } D \text{ even}.$$

Notice that $C_\pm$ defines a map $S \otimes S \to \mathbb{C}$. This map has the property of being a Spin$(V)$-morphism, so its symmetry properties can be deduced from Table 3. In terms of a basis of $S$, $\{e_\alpha\}$, and its dual, $\{e_\alpha^\vee\}$, both the morphism and the intertwining map are expressed as a matrix $C_{\pm\alpha\beta}$ called the charge conjugation matrix [5,13,14].

In the even case, $S = S^+ \oplus S^-$. For $D = 0, 4$ the morphisms $S \otimes S \to \mathbb{C}$ are block diagonal ($S^\pm \otimes S^\pm \to \mathbb{C}$), so the charge conjugation matrices must be both symmetric or both antisymmetric. For $D = 2, 6$ the morphisms are off diagonal, ($S^\pm \otimes S^\mp \to \mathbb{C}$), so the charge conjugation matrices can have simultaneously different symmetry properties. In fact, we have

$$D = 0 \bmod(8), \quad C_\pm^{\mathrm{T}} = C_\pm, \qquad D = 2 \bmod(8), \quad C_\pm^{\mathrm{T}} = \pm C_\pm,$$
$$D = 4 \bmod(8), \quad C_\pm^{\mathrm{T}} = -C_\pm, \qquad D = 6 \bmod(8), \quad C_\pm^{\mathrm{T}} = \mp C_\pm.$$

For $D$ odd we have

$$D = 1 \bmod(8), \quad C_+^{\mathrm{T}} = C_+, \qquad D = 3 \bmod(8), \quad C_-^{\mathrm{T}} = -C_-,$$
$$D = 5 \bmod(8), \quad C_+^{\mathrm{T}} = -C_+, \qquad D = 7 \bmod(8), \quad C_-^{\mathrm{T}} = C_-.$$

For arbitrary $k$ we have that the gamma matrices

$$\gamma^{[\mu_1,\dots,\mu_k]} = \frac{1}{k!}\sum_{s \in S^k}\mathrm{sig}(s)\gamma^{\mu_{s(1)}}\cdots\gamma^{\mu_{s(k)}}$$

are a map $S \to \Lambda^k \otimes S$. Composing it with $\mathcal{I} \otimes C$ we obtain a map $S \to \Lambda^k \otimes S^\vee$, which defines a map $S \otimes S \to \Lambda^k$. This map is a Spin$(V)$-morphism, and in the same basis as before is given by

$$\gamma_{\alpha\beta}^{[\mu_1,\dots,\mu_k]} = \frac{1}{k!}\sum_{s \in S^k}\mathrm{sig}(s)\gamma_\alpha^{\mu_{s(1)}\beta_1}\gamma_{\beta_1}^{\mu_{s(2)}\beta_2}\cdots\gamma_{\beta_{k-1}}^{\mu_{s(k)}\beta_k}C_{\beta_k\beta}.$$

*A note on Majorana spinors.* Consider the orthogonal group SO$(s, t)$. For $\rho_0 = 1, 7$ the spinors in the representation $S^\sigma$, of dimension $2^{(D-1)/2}$, are called Majorana spinors. For $\rho_0 = 0$ the spinors in $(S^\pm)^\sigma$ (of dimension $2^{D/2-1}$) are called Majorana–Weyl spinors. For $\rho_0 = 2, 6$ the space of Majorana spinors is $(S^+ \oplus S^-)^\sigma$, of dimension $2^{D/2}$.

For $\rho_0 = 3, 5$ the quaternionic spinors in $S$ are called pseudoMajorana spinors. For $\rho_0 = 4$, the Weyl spinors are themselves quaternionic and they are called pseudoMajorana–Weyl spinors.

The space of Majorana spinors is a real vector space and the space of pseudoMajorana spinors is a quaternionic vector space [4,5,13,14].

## 3. Poincaré supersymmetry

The Poincaré group of a space $V$ of signature $(s, t)$ is the group $\mathrm{ISO}(s, t) = \mathrm{SO}(s, t) \circledS \mathrm{T}^{s+t}$. We consider super Poincaré algebras with non-extended supersymmetry ($N = 1$). The anti-commutator of the odd generators (spinor charges) is in the representation $\mathrm{Sym}(S \otimes S)$. One can decompose it into irreducible representations under the group $\mathrm{Spin}(V)$. It is a fact that only antisymmetric tensor representations will appear. Poincaré supersymmetry requires the presence of the vector representation in this decomposition to accommodate the momenta $P_\mu$. Another way of expressing this is by saying that there must be a morphism

$$S \otimes S \to V,$$

which is symmetric. This can be read from Table 3. In the table, complex representations are considered. Since the Poincaré superalgebra is a real superalgebra, care should be exercised when interpreting it in the different cases of real, quaternionic and complex spinors. We will deal separately with these cases.

*Real case.* The most general form of the anticommutator of two spinor generators is

$$\{Q_\alpha, Q_\beta\} = \sum_k \gamma^{[\mu_1 \cdots \mu_k]}_{(\alpha\beta)} Z_{[\mu_1 \cdots \mu_k]}, \tag{3}$$

where $Z_{[\mu_1 \cdots \mu_k]}$ are even generators. In the sum there appear only the terms that are symmetric with respect to $\alpha$ and $\beta$; we indicate it by $(\alpha\beta)$.

If the term $\gamma^\mu_{(\alpha\beta)}$ appears, then a super Poincaré algebra exists. The rest of the $Z$ generators can be taken to commute among themselves and with the odd generators and transform appropriately with the Lorentz generators. We have then the maximal "central extension" [1] of the super Poincaré algebra.

For $\rho_0 = 0$, since the Weyl spinors are real one can have a chiral superalgebra. The vector representation should appear then in the symmetric product $\mathrm{Sym}(S^\pm \otimes S^\pm)$. This happens only for $D_0 = 2$ ($\rho$ and $D$ have the same parity). If we consider non-chiral superalgebras, where both $S^\pm$ are present, also the values $D_0 = 0, 4$ are allowed.

For $\rho_0 = 1, 7$, we have $D_0 = 1, 3$.

*Quaternionic case.* The most general anticommutator of two spinor charges is

$$\{Q^i_\alpha, Q^j_\beta\} = \sum_k \gamma^{[\mu_1 \cdots \mu_k]}_{[\alpha\beta]} Z^0_{[\mu_1 \cdots \mu_k]} \Omega^{ij} + \sum_k \gamma^{[\mu_1 \cdots \mu_k]}_{(\alpha\beta)} Z^I_{[\mu_1 \cdots \mu_k]} \sigma^{ij}_I. \tag{4}$$

$\sigma^{ij}_I$ are the (symmetric) Pauli matrices, $i, j = 1, 2$, $I = 1, 2, 3$. (We have multiplied them by the invariant antisymmetric metric $\Omega^{ij}$). If we demand that the momentum $P_\mu$ is a singlet under the full R-symmetry group $\mathrm{SU}(2) \simeq \mathrm{Usp}(2)$, then the $\gamma^\mu_{\alpha\beta}$ must be antisymmetric and the momentum appears in the first term (singlet) of the r.h.s. of (4).

For $\rho_0 = 3, 5$, this happens if $D_0 = 5, 7$. The only even case is $\rho_0 = 4$. A chiral superalgebra exists for $D_0 = 6$, while for $D_0 = 0, 4$ one can have non-chiral superalgebras.

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[1] Except for $k = 0$, the generators $Z$ are not central elements, since they do not commute with the elements of the Lorentz group. They are central only in the super translation algebra. It is nevertheless customary to call them "central charges".

If we restrict the R-symmetry group to $SO^*(2)$, there is also an invariant symmetric metric, $\delta^{ij}$. In the anticommutator (4) we can consider terms like

$$\sum_k \gamma_{(\alpha\beta)}^{[\mu_1\cdots\mu_k]} Z_{[\mu_1\cdots\mu_k]}\delta^{ij}.$$

The $\gamma_{\alpha\beta}^\mu$ must be symmetric to appear in such term. For $\rho_0 = 3, 5$, this happens if $D_0 = 1, 3$. For $\rho_0 = 4$ and $D_0 = 2$ a chiral superalgebra exists.

For $\rho_0 = 4$ and $D_0 = 0, 4$ one can have non chiral superalgebras.

*Complex case.* This is the case for $\rho_0 = 2, 6$. The spinor charges are in the representation $S^+ \oplus S^-$ and we will denote them by $(Q_\alpha, Q_{\dot\alpha})$. In the anticommutator there are three pieces,

$$\{Q_\alpha, Q_\beta\}, \qquad \{Q_{\dot\alpha}, Q_{\dot\beta}\}, \qquad \{Q_\alpha, Q_{\dot\alpha}\},$$

and it is clear that only the last one is invariant under the R-symmetry group $U(1)$. Then there must be a morphism

$$S^+ \otimes S^- \to \Lambda^1.$$

This happens in the cases $D_0 = 0, 4$.

We summarize these results in Table 4. The values of $m, n$ are such that $s, t \geq 0$. We mark with "†" the non-chiral superalgebras. We note that for standard space–time signature,

Table 4
Poincaré groups with supersymmetric extensions

| $(D_0, \rho_0)$ | $ISO(s, t)$ |
| --- | --- |
| $(2, 0)$ | $ISO(1 + 8n + 4m, 1 + 4m)$ |
| $(0, 2)$ | $ISO(1 + 8n + 4m, -1 + 4m)$ |
| $(4, 2)$ | $ISO(3 + 8n + 4m, 1 + 4m)$ |
| $(2, 4)$ | $ISO(3 + 8n + 4m, -1 + 4m)$ |
| $(6, 4)$ | $ISO(5 + 8n + 4m, 1 + 4m)$ |
| $(0, 6)$ | $ISO(3 + 8n + 4m, -3 + 4m)$ |
| $(4, 6)$ | $ISO(9 + 8n + 4m, 3 + 4m)$ |
| $(0, 0)^\dagger$ | $ISO(8n + 4m, 4m)$ |
| $(0, 4)^\dagger$ | $ISO(2 + 8n + 4m, -2 + 4m)$ |
| $(4, 0)^\dagger$ | $ISO(2 + 8n + 4m, 2 + 4m)$ |
| $(4, 4)^\dagger$ | $ISO(4 + 8n + 4m, 4m)$ |
| $(1, 1)$ | $ISO(1 + 8n + 4m, 4m)$ |
| $(1, 3)$ | $ISO(2 + 8n + 4m, -1 + 4m)$ |
| $(3, 3)$ | $ISO(3 + 8n + 4m, 4m)$ |
| $(1, 5)$ | $ISO(3 + 8n + 4m, -1 + 4m)$ |
| $(3, 5)$ | $ISO(4 + 8n + 4m, -1 + 4m)$ |
| $(3, 1)$ | $ISO(2 + 8n + 4m, 1 + 4m)$ |
| $(5, 3)$ | $ISO(4 + 8n + 4m, 1 + 4m)$ |
| $(7, 3)$ | $ISO(5 + 8n + 4m, 2 + 4m)$ |
| $(5, 5)$ | $ISO(5 + 8n + 4m, 4m)$ |
| $(7, 5)$ | $ISO(6 + 8n + 4m, 1 + 4m)$ |
| $(1, 7)$ | $ISO(4 + 8n + 4m, -3 + 4m)$ |
| $(3, 7)$ | $ISO(5 + 8n + 4m, -2 + 4m)$ |

$\text{ISO}(D - 1, 1) = \text{ISO}(\rho + 1 + 8n, 1)$, all super Poincaré algebras are present for $D = 0, \dots, 7 \bmod(8)$.

## 4. Conformal supersymmetry

The conformal group of a vector space $V$ of signature $(s - 1, t - 1)$ is the group of coordinate transformations that leave the metric invariant up to a scale change. This group is isomorphic to $\text{SO}(s, t)$, a simple group, for $D \geq 3$. The Poincaré group $\text{ISO}(s, t)$ is a subgroup of the conformal group. In a space with the standard Minkowski signature $(s - 1, 1)$, the conformal group is the simple group $\text{SO}(s, 2)$. It is also the anti de Sitter group in dimension $s + 1$.

A simple superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ satisfies necessarily

$$\{\mathcal{A}_1, \mathcal{A}_1\} = \mathcal{A}_0. \tag{5}$$

We look for minimal simple superalgebras (with minimal number of even generators) containing space–time conformal symmetry in its even part. The odd generators are in a spinor representation $S$ of $\text{Spin}(s, t)$, and all the even generators should appear in the r.h.s. of the anticommutator of the spinor charges, which is in the $\text{Sym}(S \otimes S)$ representation. As we did in the case of Poincaré supersymmetry, we decompose it with respect to $\text{Spin}(s, t)$. The orthogonal generators are in the antisymmetric 2-fold representation, so we should look for morphisms

$$S \otimes S \to \Lambda^2$$

with the appropriate symmetry properties for each signature and dimension. The discussion is as for Poincaré supersymmetry, but with $k = 2$ in Table 3.

For the real case the matrices should be symmetric. We have $\rho_0 = 0$ with $D_0 = 4$ and $\rho_0 = 1, 7$ with $D_0 = 3, 5$. For the quaternionic case the matrices should be antisymmetric if we demand invariance under the SU(2) R-symmetry. We have $\rho_0 = 4$ with $D_0 = 8$ and $\rho_0 = 3, 5$ with $D_0 = 1, 7$. If the R-symmetry is restricted to $\text{SO}^*(2)$ the singlet is $\delta^{ij}$, while the $\text{SO}^*(2)$ generator is $\Omega^{ij}$. Then we have $\rho_0 = 4$ with $D_0 = 4$ and $\rho_0 = 3, 5$ with $D_0 = 3, 5$.

For the complex case, if we demand that the orthogonal generators are singlets under U(1), the matrices should be in $S^+ \otimes S^-$, which is invariant under the U(1) R-symmetry group. We have $\rho_0 = 2, 6$ with $D_0 = 2, 6$. For $\rho_0 = 0$ and $D_0 = 2, 6$ we have a superalgebra containing the orthogonal group in its even part provided we take two spinors, one in $S^+$ and the other in $S^-$. $\rho_0 = 4$ and $D_0 = 2, 6$ is a similar case, but the spinors in $S^{\pm}$ should have also an SU(2) index. We may also consider the cases $\rho_0 = 2, 6$ and $D_0 = 4$ where the U(1) invariance is not present. Then, the orthogonal generators are in the anticommutator $\text{Sym}(S^+ \otimes S^+)$.

When the morphism is such that the orthogonal generators are in the r.h.s. of the anti-commutator of the odd generators, the biggest simple group that one can consider is the one generated by all the symmetric matrices. This is the symplectic group $\text{Sp}(2n, \mathbb{R})$ where $2n$ is the real dimension of the spinor charge. As we will see there is a superalgebra with bosonic part $\text{sp}(2n, \mathbb{R})$, one of the orthosymplectic algebras. In the quaternionic case, we observe

Table 5
Orthogonal groups and their symplectic embeddings

| $(D_0, \rho_0)$ | SO$(s, t)$ | Sp$(2n, \mathbb{R})$ |
|---|---|---|
| $(0, 4)$ | SO$(2 + 8n + 4m, -2 + 4m) \times$ SU$(2)$ | Sp$(2^{4(n+m)})$ |
| $(2, 0)^{\dagger}$ | SO$(1 + 8n + 4m, 1 + 4m)$ | Sp$(2^{1+4(n+m)})$ |
| $(2, 2)$ | SO$(2 + 8n + 4m, 4m) \times$ U$(1)$ | Sp$(2^{1+4(n+m)})$ |
| $(2, 4)^{\dagger}$ | SO$(3 + 8n + 4m, -1 + 4m) \times$ SU$(2)$ | Sp$(2^{2+4(n+m)})$ |
| $(2, 6)$ | SO$(4 + 8n + 4m, -2 + 4m) \times$ U$(1)$ | Sp$(2^{1+4(n+m)})$ |
| $(4, 0)$ | SO$(2 + 8n + 4m, 2 + 4m)$ | Sp$(2^{1+4(n+m)})$ |
| $(4, 2)$ | SO$(3 + 8n + 4m, 1 + 4m)$ | Sp$(2^{2+4(n+m)})$ |
| $(4, 4)$ | SO$(4 + 8n + 4m, 4m) \times$ SO$^*(2)$ | Sp$(2^{2+4(n+m)})$ |
| $(4, 6)$ | SO$(5 + 8n + 4m, -1 + 4m)$ | Sp$(2^{2+4(n+m)})$ |
| $(6, 0)^{\dagger}$ | SO$(3 + 8n + 4m, 3 + 4m)$ | Sp$(2^{1+4(n+m)})$ |
| $(6, 2)$ | SO$(4 + 8n + 4m, 2 + 4m) \times$ U$(1)$ | Sp$(2^{3+4(n+m)})$ |
| $(6, 4)^{\dagger}$ | SO$(5 + 8n + 4m, 1 + 4m) \times$ SU$(2)$ | Sp$(2^{2+4(n+m)})$ |
| $(6, 6)$ | SO$(6 + 8n + 4m, 4m) \times$ U$(1)$ | Sp$(2^{3+4(n+m)})$ |
| $(1, 3)$ | SO$(2 + 8n + 4m, -1 + 4m) \times$ SU$(2)$ | Sp$(2^{1+4(n+m)})$ |
| $(1, 5)$ | SO$(3 + 8n + 4m, -2 + 4m) \times$ SU$(2)$ | Sp$(2^{1+4(n+m)})$ |
| $(3, 1)$ | SO$(2 + 8n + 4m, 1 + 4m)$ | Sp$(2^{5+4(n+m)})$ |
| $(3, 3)$ | SO$(3 + 8n + 4m, 4m) \times$ SO$^*(2)$ | Sp$(2^{2+4(n+m)})$ |
| $(3, 5)$ | SO$(4 + 8n + 4m, -2 + 4m) \times$ SO$^*(2)$ | Sp$(2^{2+4(n+m)})$ |
| $(3, 7)$ | SO$(5 + 8n + 4m, -2 + 4m)$ | Sp$(2^{1+4(n+m)})$ |
| $(5, 1)$ | SO$(3 + 8n + 4m, 2 + 4m)$ | Sp$(2^{2+4(n+m)})$ |
| $(5, 3)$ | SO$(4 + 8n + 4m, 2 + 4m) \times$ SO$^*(2)$ | Sp$(2^{3+4(n+m)})$ |
| $(5, 5)$ | SO$(5 + 8n + 4m, 4m) \times$ SO$^*(2)$ | Sp$(2^{3+4(n+m)})$ |
| $(5, 7)$ | SO$(6 + 8n + 4m, -1 + 4m)$ | Sp$(2^{2+4(n+m)})$ |
| $(7, 3)$ | SO$(5 + 8n + 4m, 2 + 4m) \times$ SU$(2)$ | Sp$(2^{4+4(n+m)})$ |
| $(7, 5)$ | SO$(6 + 8n + 4m, 1 + 4m) \times$ SU$(2)$ | Sp$(2^{4+4(n+m)})$ |

that if the morphism to $\Lambda^2$ is antisymmetric (symmetric), then the morphism to $\Lambda^0 = \mathbb{C}$ is symmetric (antisymmetric). It follows from (4) that in this case the orthogonal group times SU(2) is a subgroup of the symplectic group. In the complex case the orthogonal group will come multiplied by U(1) (unless $D_0 = 4$). In these cases, SU(2) and U(1) are groups of automorphisms of the supersymmetry algebra.

The results are summarized in Table 5 with the same conventions as in Table 4. We mark with "†" the cases that lead to non-chiral superalgebras.

The case of SO(2, 2) would naively correspond to an embedding in Sp(2, $\mathbb{R}$). This is obviously not true and the reason is that O(2, 2) is not simple, so property (5) does not hold. In fact, since SO(2, 2) $\simeq$ SO(2, 1) $\times$ SO(2, 1), we have that Sp(2, $\mathbb{R}$) $\simeq$ SO(2, 1), one of the simple factors.

## 5. The orthosymplectic algebra and space–time supersymmetry

We recall here the definition of the orthosymplectic superalgebra osp$(N|2p, \mathbb{R})$ [11,17,30]. Consider the $\mathbb{Z}_2$-graded vector space $E = V \oplus H$, with dim$(V) = N$ and dim$(H) = 2p$. End$(E)$ is a super Lie algebra in the usual way, with the even part End$(E)_0 =$

$\mathrm{End}(V) \oplus \mathrm{End}(H)$. In terms of an homogeneous basis, an element of $\mathrm{End}(E)$ is a real matrix

$$\begin{pmatrix} A_{N \times N} & B_{2p \times N} \\ C_{N \times 2p} & D_{2p \times 2p} \end{pmatrix}.$$

Then

$$\mathrm{End}(E)_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad \mathrm{End}(E)_1 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

with the usual bracket

$$[a, b] = ab - (-1)^{g(a)g(b)} ba, \quad a, b \in \mathrm{End}(E). \tag{6}$$

($g = 0, 1$ will denote the grading on both spaces, $E$ and $\mathrm{End}(E)$).

Consider on $E$ a non-degenerate bilinear form $F$ such that $F(u, v) = (-1)^{p(u)p(v)} F(v, u)$ and $F(u, v) = 0$ for $u \in E_0$, $v \in E_1$. Then, there exists an homogeneous basis where

$$F = \begin{pmatrix} \Omega_{N \times N} & 0 \\ 0 & \Omega_{2p \times 2p} \end{pmatrix}$$

with

$$\Omega_{2p \times 2p}^2 = -\mathcal{I}, \qquad \Omega_{2p \times 2p}^{\mathrm{T}} = -\Omega_{2p \times 2p}, \qquad \Omega_{N \times N}^{\mathrm{T}} = \Omega_{N \times N}.$$

The orthosymplectic algebra $\mathrm{osp}(N|2p, \mathbb{R})$ is the set of real $(N + 2p) \times (N + 2p)$ matrices $a$ satisfying

$$a^{\mathrm{T}} F + Fa = 0$$

with bracket (6). The even part is $\mathrm{so}(N) \oplus \mathrm{sp}(2p)$, and the generators of the odd part are in the fundamental representation $(N, 2p)$. It is a simple superalgebra, so in particular,

$$\{\mathrm{osp}(N|2p, \mathbb{R})_1, \mathrm{osp}(N|2p, \mathbb{R})_1\} = \mathrm{so}(N) \oplus \mathrm{sp}(2p, \mathbb{R}). \tag{7}$$

Given the results of Section 4, the orthosymplectic superalgebras are the supersymmetric extensions of the conformal group of space–time. We take $N = 1$, and $2p$ according to Table 5. The defining representation of the symplectic group is the corresponding spinor representation of the orthogonal subgroup.

The symplectic algebra $\mathrm{sp}(2p)$ has a maximal subalgebra $\mathrm{sl}(p, \mathbb{R}) \oplus \mathrm{so}(1, 1)$. The fundamental representation of $\mathrm{sp}(2p)$ decomposes as

$$\mathbf{2p} \underset{\mathrm{sl}(p,\mathbb{R}) \oplus \mathrm{so}(1,1)}{\rightarrow} (\mathbf{p}, \tfrac{1}{2}) \oplus (\mathbf{p}', -\tfrac{1}{2}),$$

where $\mathbf{p}'$ is the dual representation to $\mathbf{p}$. The decomposition of the adjoint representation is

$$\mathrm{Sym}(\mathbf{2p} \otimes \mathbf{2p}) \underset{\mathrm{sl}(p,\mathbb{R}) \oplus \mathrm{so}(1,1)}{\rightarrow} (\mathrm{Sym}(\mathbf{p} \otimes \mathbf{p}), 1) \oplus (\mathrm{adj}_{\mathrm{sl}(p)}, 0) \oplus (1, 0)$$
$$\oplus (\mathrm{Sym}(\mathbf{p}' \otimes \mathbf{p}'), -1).$$

This defines an so(1, 1), Lie algebra grading of $\mathcal{L} = \mathrm{sp}(2p)$

$$\mathcal{L}_{\mathrm{sp}} = \mathcal{L}_{\mathrm{sp}}^{+1} \oplus \mathcal{L}_{\mathrm{sp}}^{0} \oplus \mathcal{L}_{\mathrm{sp}}^{-1}, \tag{8}$$

where the superindices are the so(1, 1) charges. The direct sums here are understood as vector space sums, not as Lie algebra sums. By the properties of the grading, $\mathcal{L}_{\mathrm{sp}}^{+1}$ and $\mathcal{L}_{\mathrm{sp}}^{-1}$ are Abelian subalgebras.

The orthogonal group SO(s, t) contains as a subgroup ISO(s − 1, t − 1). In the algebra, the adjoint of so(s, t) contains a singlet under SO(s − 1, t − 1). The corresponding so(1, 1)-grading is like (8), and in fact they coincide when the orthogonal algebra is seen as a subalgebra of the symplectic one. For the orthogonal case we have

$$\mathcal{L}_{\mathrm{o}}^{+1} = \{P_\mu\}, \qquad \mathcal{L}_{\mathrm{o}}^{0} = \mathrm{so}(s-1, t-1) \oplus \mathrm{o}(1, 1), \qquad \mathcal{L}_{\mathrm{o}}^{-1} = \{K_\mu\},$$

where $P_\mu$ and $K_\mu$ are so(s − 1, t − 1) vectors satisfying

$$[P_\mu, P_\nu] = [K_\mu, K_\nu] = 0.$$

$\mathcal{L}_{\mathrm{o}}^{0}$ contains the orthogonal generators $M_{\mu\nu} \in \mathrm{so}(s-1, t-1)$ and the dilatation $D$. $P_\mu$ can be identified with the momenta of ISO(s − 1, t − 1), and $K_\mu$ are the conformal boost generators.

When we consider the supersymmetric extension of Sp(2p) as the orthosymplectic algebra osp(1|2p, ℝ), the previous grading is extended and we have the decomposition

$$\mathcal{L}_{\mathrm{osp}} = \mathcal{L}_{\mathrm{osp}}^{+1} \oplus \mathcal{L}_{\mathrm{osp}}^{+1/2} \oplus \mathcal{L}_{\mathrm{osp}}^{0} \oplus \mathcal{L}_{\mathrm{osp}}^{-1/2} \oplus \mathcal{L}_{\mathrm{osp}}^{-1},$$

where $\mathcal{L}_{\mathrm{osp}}^{\pm 1/2}$ contains the odd generators of the superalgebra, which are in the fundamental representation of sp(2p). This representation decomposes as

$$\mathbf{2p} \underset{\mathrm{sl}(p, \mathbb{R}) \oplus \mathrm{so}(1,1)}{\longrightarrow} (\mathbf{p}, +\tfrac{1}{2}) \oplus (\mathbf{p}, -\tfrac{1}{2}),$$

so

$$\mathcal{L}_{\mathrm{osp}}^{+(1/2)} = \{Q_\alpha\}, \qquad \mathcal{L}_{\mathrm{osp}}^{-(1/2)} = \{S_\alpha\}.$$

It is important to remark that since the signature $\rho$ is the same for so(s, t) and so(s − 1, t − 1), the spinors have the same reality properties. Furthermore, the irreducible spinor of so(s, t) ⊂ sp(2p) decomposes into two irreducible spinors of so(s − 1, t − 1) ⊂ sl(p) with opposite grading. These are usually denoted as the $Q$ and $S$ spinors of the super conformal algebra.

When $D = s + t$ is even, the irreducible spinor $S^\pm$ of so(s, t) decomposes into two spinors of so(s − 1, t − 1) of opposite chiralities,

$$S_D^+ \rightarrow Q_{D-2}^+ \oplus S_{D-2}^-$$

(the superindex here indicates chirality, not the so(1, 1)-grading). Since one has the morphism [28]

$$Q_{D-2}^\pm \otimes V \rightarrow S_{D-2}^\mp,$$

then the commutator of a charge of a certain chirality with $K_\mu$ or $P_\mu$ must give a charge with the opposite chirality.

The subalgebra $\mathcal{L}^{+1} \oplus \mathcal{L}^{+(1/2)}$ is a nilpotent subalgebra of $\mathrm{osp}(1|2p, \mathbb{R})$, which in fact is the maximal central extension of the super translation algebra. The full set of central charges transforms, therefore, in the symmetric representation of $\mathrm{sl}(p, \mathbb{R})$ while the odd charges transform in the fundamental representation of the same group. We observe that the orthosymplectic algebra has twice the number of odd generators than the super Poincaré algebra.

## 5.1. Contractions of the orthosymplectic algebra

The Poincaré algebra can also be obtained from an orthogonal algebra by an Inonü–Wigner contraction

$$\mathrm{so}(s, t+1) \underset{\text{contraction}}{\rightarrow} \mathrm{iso}(s, t),$$

as a generalization of the well-known case of the anti-de-Sitter group in $D - 1$ dimensions

$$\mathrm{AdS}_{D-1} = \mathrm{SO}(D-2, 2) \underset{\text{contraction}}{\rightarrow} \mathrm{ISO}(D-2, 1).$$

In fact, the same Poincaré algebra can be obtained also by the contraction

$$\mathrm{so}(s+1, t) \underset{\text{contraction}}{\rightarrow} \mathrm{iso}(s, t).$$

The contraction is defined as follows. Let $T_{[AB]}$ be the generators of $\mathrm{so}(s, t+1)$ or $\mathrm{so}(s+1, t)$, $A, B = 1, \ldots, D'$, $D' = s + t + 1$. Let $\mu, \nu = 1, \ldots, D$ and consider the decomposition $T_{[\mu\nu]}, T_{[\mu, D']}$. We define $T'_\mu = (1/e)T_{[\mu, D']}$ and take the limit $e \to 0$ in the algebra while keeping finite the generators $T_{[\mu\nu]}, T_{[\mu, D']}$. The result is the algebra of the Poincaré group $\mathrm{ISO}(s, t)$ with $P_\mu = T_{[\mu, D']}$.

We consider now the following contraction of the orthosymplectic superalgebra. The generators of the bosonic subalgebra $Z_{[\mu_1 \cdots \mu_k]}$ appear in the r.h.s. of (9)

$$\{Q_\alpha, Q_\beta\} = \sum_k \gamma_{\alpha\beta}^{[\mu_1 \cdots \mu_k]} Z_{[\mu_1 \cdots \mu_k]}, \quad \mu_i = 1, \ldots, D. \tag{9}$$

(Only the $\gamma$'s with the appropriate symmetry will appear.) We set

$$Z_{[\mu_1 \cdots \mu_k]} \to \frac{1}{e} Z_{[\mu_1 \cdots \mu_k]}, \qquad Q \to \frac{1}{\sqrt{e}} Q.$$

We obtain a superalgebra with bosonic part totally Abelian.

Consider a symplectic group containing an orthogonal group in dimension $D$ and signature $\rho$ according to Table 5, and the contraction of the orthosymplectic algebra as explained above. We can decompose the odd and even generators with respect to the orthogonal subgroup $(D-1, \rho+1)$ or $(D-1, \rho-1)$. Interpreting $Z_{[\mu D]}$ as the momentum in dimension $D - 1$ ($\mu$ taken only from 1 to $D - 1$), the algebra is then seen to be the maximal central extension of the super translation algebra in $(D-1, \rho \pm 1)$.

If all the symplectic generators are contracted except the generators of the orthogonal group,

$$Z_{[\mu\nu]} \mapsto Z_{[\mu\nu]}, \quad \mu, \nu = 1, \ldots, D-1,$$

then one obtains a super Poincaré algebra. It has "central extension", but it is not maximal since the generators of the orthogonal group $SO(s, t)$, $Z_{[\mu\nu]}$, are not commuting and do not appear in the r.h.s. of (9).

The spinor representations of the orthogonal group in $(D, \rho)$ behave differently when decomposing with respect to the orthogonal subgroup in $(D-1, \rho \pm 1)$, depending on $\rho$. For the complex spinors we have that for $D \to D-1$

$$S_D^\pm \mapsto S_{D-1}, \quad D \text{ even}, \qquad S_D \mapsto S_{D-1}^+ \oplus S_{D-1}^-, \quad D \text{ odd}.$$

Over the reals, the representation may or may not remain irreducible. We make the analysis first for $(D-1, \rho+1)$. The representation remains irreducible for $\rho_0 = 0, 1, 2, 4$ while for $\rho_0 = 3, 5, 6, 7$ it splits into two spinor representations, so the super Poincaré algebra obtained has $N = 2$ supersymmetry. More precisely, for $\rho_0 = 3, 7$ we get two spinors of different chirality, so we have (1, 1) supersymmetry, while for $\rho = 5, 6$ we get $N = 2$ supersymmetry.

The orthogonal group for $(D-1, \rho-1)$ is isomorphic to the orthogonal group for $(D-1, -\rho+1)$, so the decomposition of the representations under $\rho \mapsto \rho-1$ can be formulated as a decomposition of the type $\rho' \mapsto \rho'+1$ with $\rho' = -\rho$. It is then enough to write the decompositions $\rho \mapsto \rho+1$. We give them in Table 6.

We can now apply these decompositions to the list given in Table 5. The super Poincaré algebra for $(D, \rho)$ could be in principle obtained by contraction from two different orthosymplectic algebras, the ones corresponding to orthogonal groups $(D+1, \rho+1)$ or $(D+1, \rho-1)$. However, it may happen that no one of them exists, as in $(D_0, \rho_0) = (0, 0), (4, 4)$ or that only one exists, as for $D_0 = 3, 7$, $\rho_0 = 3, 5$ and $D_0 = 0, 2, 4, 6$, $\rho_0 = 2, 6$. The rest have both possibilities.

The orthosymplectic algebra corresponding to $(D, \rho)$ can be contracted in two different ways, as the bosonic orthogonal algebra. However these contractions do not lead necessarily to one of the Poincaré superalgebras listed in Table 4, since we imposed some restrictions on the algebras appearing in that table.

Table 6
Decomposition of spinors $(\rho, D) \to (\rho+1, D-1)$

| $\rho_0$ | $(\rho, D) \to (\rho+1, D-1)$ | Reducibility |
|---|---|---|
| 0 | $S_\mathbb{R}^+ \to S_\mathbb{R}$ | Irreducible |
| 1 | $S_\mathbb{R} \to S_\mathbb{C}^+ \oplus \overline{S_\mathbb{C}^+}$ | Irreducible |
| 2 | $S_\mathbb{C}^+ \to S_\mathbb{H}$ | Irreducible |
| 3 | $S_\mathbb{H} \to S_\mathbb{H}^+ \oplus S_\mathbb{H}^-$ | Reducible |
| 4 | $S_\mathbb{H}^+ \to S_\mathbb{H}$ | Irreducible |
| 5 | $S_\mathbb{H} \to S_\mathbb{C}^+ \oplus \overline{S_\mathbb{C}^-}$ | Reducible |
| 6 | $S_\mathbb{C}^+ \to S_\mathbb{R} \oplus S_\mathbb{R}$ | Reducible |
| 7 | $S_\mathbb{R} \to S_\mathbb{R}^+ \oplus S_\mathbb{R}^-$ | Reducible |

Let us note that in particular, the Poincaré algebras corresponding to the physically interesting case, $\rho = D - 2$, are all obtained by contraction. For $D = 8, 9$ the algebra obtained has extended ($N = 2$) supersymmetry. For $D = 6, 10$ the algebras obtained are non-chiral.

## 6. Orthogonal, symplectic and linear spinors

We consider now morphisms [29,39]

$$S \otimes S \to \Lambda^0 \simeq \mathbb{C}.$$

If a morphism of this kind exists, it is unique up to a multiplicative factor. The vector space of the spinor representation has then a bilinear form invariant under $\mathrm{Spin}(V)$. Looking at Table 3, one can see that this morphism exists except for $D_0 = 2, 6$, where instead a morphism

$$S^{\pm} \otimes S^{\mp} \to \mathbb{C}$$

occurs.

We shall call a spinor representation orthogonal if it has a symmetric, invariant bilinear form. This happens for $D_0 = 0, 1, 7$ and $\mathrm{Spin}(V)^{\mathbb{C}}$ (complexification of $\mathrm{Spin}(V)$) is then a subgroup of the complex orthogonal group $\mathrm{O}(n, \mathbb{C})$, where $n$ is the dimension of the spinor representation (Weyl spinors for $D$ even). The generators of $\mathrm{O}(n, \mathbb{C})$ are $n \times n$ antisymmetric matrices. These are obtained in terms of the morphisms

$$S \otimes S \to \Lambda^k,$$

which are antisymmetric. This gives the decomposition of the adjoint representation of $\mathrm{O}(n, \mathbb{C})$ under the subgroup $\mathrm{Spin}(V)^{\mathbb{C}}$. In particular, for $k = 2$ one obtains the generators of $\mathrm{Spin}(V)^{\mathbb{C}}$.

A spinor representation is called symplectic if it has an antisymmetric, invariant bilinear form. This is the case for $D_0 = 3, 4, 5$. $\mathrm{Spin}(V)^{\mathbb{C}}$ is a subgroup of the symplectic group $\mathrm{Sp}(2p, \mathbb{C})$, where $2p$ is the dimension of the spinor representation. The Lie algebra $\mathrm{sp}(2p, \mathbb{C})$ is formed by all the symmetric matrices, so it is given in terms of the morphisms $S \otimes S \to \Lambda^k$ which are symmetric. The generators of $\mathrm{Spin}(V)^{\mathbb{C}}$ correspond to $k = 2$ and are symmetric matrices.

For $D_0 = 2, 6$ one has an invariant morphism

$$B : S^{+} \otimes S^{-} \to \mathbb{C}.$$

The representations $S^{+}$ and $S^{-}$ are the contragradient (or dual) of one another. The spin representations extend to representations of the linear group $\mathrm{GL}(n, \mathbb{C})$, which leaves the pairing $B$ invariant. These spinors are called linear. $\mathrm{Spin}(V)^{\mathbb{C}}$ is a subgroup of the simple factor $\mathrm{SL}(n, \mathbb{C})$.

These properties depend exclusively on the dimension. When combined with the reality properties, which depend on $\rho$, one obtains real groups embedded in $\mathrm{SO}(n, \mathbb{C})$, $\mathrm{Sp}(2p, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C})$ which have an action on the space of the real spinor representation $S^{\sigma}$. The real groups contain as a subgroup $\mathrm{Spin}(V)$.

We need first some general facts about real forms of simple Lie algebras. Let $S$ be a complex vector space of dimension $n$ which carries an irreducible representation of a complex Lie algebra $\mathcal{G}$. Let $G$ be the complex Lie group associated to $\mathcal{G}$. Let $\sigma$ be a conjugation or a pseudoconjugation on $S$ such that $\sigma X \sigma^{-1} \in \mathcal{G}$ for all $X \in \mathcal{G}$. Then the map

$$X \mapsto X^\sigma = \sigma X \sigma^{-1}$$

is a conjugation of $\mathcal{G}$. We shall write

$$\mathcal{G}^\sigma = \{X \in \mathcal{G} | X^\sigma = X\}.$$

$\mathcal{G}^\sigma$ is a real form of $\mathcal{G}$. If $\tau = h\sigma h^{-1}$, with $h \in G$, $\mathcal{G}^\tau = h\mathcal{G}^\sigma h^{-1}$. $\mathcal{G}^\sigma = \mathcal{G}^{\sigma'}$ if and only if $\sigma' = \epsilon\sigma$ for $\epsilon$ a scalar with $|\epsilon| = 1$; in particular, if $\mathcal{G}^\sigma$ and $\mathcal{G}^\tau$ are conjugate by $G$, $\sigma$ and $\tau$ are both conjugations or both pseudoconjugations. The conjugation can also be defined on the group $G$, $g \mapsto \sigma g \sigma^{-1}$.

### 6.1. Real forms of the classical Lie algebras

We describe the real forms of the classical Lie algebras from this point of view. (see also [29]).

*Linear algebra*, sl(S).

1. If $\sigma$ is a conjugation of $S$, then there is an isomorphism $S \to \mathbb{C}^n$ such that $\sigma$ goes over to the standard conjugation of $\mathbb{C}^n$. Then $\mathcal{G}^\sigma \simeq \mathrm{sl}(n, \mathbb{R})$. (The conjugation acting on $\mathrm{gl}(n, \mathbb{C})$ gives the real form $\mathrm{gl}(n, \mathbb{R})$.)
2. If $\sigma$ is a pseudoconjugation and $\mathcal{G}$ does not leave invariant a non-degenerate bilinear form, then there is an isomorphism of $S$ with $\mathbb{C}^n$, $n = 2p$ such that $\sigma$ goes over to

$$(z_1, \ldots, z_p, z_{p+1}, \ldots, z_{2p}) \mapsto (z_{p+1}^*, \ldots, z_{2p}^*, -z_1^*, \ldots, -z_p^*).$$

   Then $\mathcal{G}^\sigma \simeq \mathrm{su}^*(2p)$. (The pseudoconjugation acting in on $\mathrm{gl}(2p, \mathbb{C})$ gives the real form $\mathrm{su}^*(2p) \oplus \mathrm{so}(1, 1)$.)
   To see this, it is enough to prove that $\mathcal{G}^\sigma$ does not leave invariant any non-degenerate Hermitian form, so it cannot be of the type $\mathrm{su}(p, q)$. Suppose that $\langle \cdot, \cdot \rangle$ is a $\mathcal{G}^\sigma$-invariant, non-degenerate Hermitian form. Define $(s_1, s_2) := \langle \sigma(s_1), s_2 \rangle$. Then $(\cdot, \cdot)$ is bilinear and $\mathcal{G}^\sigma$-invariant, so it is also $\mathcal{G}$-invariant.
3. The remaining cases, following Cartan's classification of real forms of simple Lie algebras, are $\mathrm{su}(p, q)$, where a non-degenerate Hermitian bilinear form is left invariant. They do not correspond to a conjugation or pseudoconjugation on $S$, the space of the fundamental representation. (The real form of $\mathrm{gl}(n, \mathbb{C})$ is in this case $\mathrm{u}(p, q)$.)

*Orthogonal algebra*, so(S). $\mathcal{G}$ leaves invariant a non-degenerate, symmetric bilinear form. We will denote it by $(\cdot, \cdot)$.

1. If $\sigma$ is a conjugation preserving $\mathcal{G}$, one can prove that there is an isomorphism of $S$ with $\mathbb{C}^n$ such that $(\cdot, \cdot)$ goes to the standard form and $\mathcal{G}^\sigma$ to $\mathrm{so}(p, q)$, $p + q = n$. Moreover, all $\mathrm{so}(p, q)$ are obtained in this form.

2. If $\sigma$ is a pseudoconjugation preserving $\mathcal{G}$, $\mathcal{G}^\sigma$ cannot be any of the so$(p, q)$. By Cartan's classification, the only other possibility is that $\mathcal{G}^\sigma \simeq$ so$^*(2p)$. There is an isomorphism of $S$ with $\mathbb{C}^{2p}$ such that $\sigma$ goes to

$$(z_1, \ldots, z_p, z_{p+1}, \ldots, z_{2p}) \mapsto (z_{p+1}^*, \ldots, z_{2p}^*, -z_1^*, \ldots, -z_p^*).$$

*Symplectic algebra*, sp(S). We denote by $(\cdot, \cdot)$ the symplectic form on $S$.

1. If $\sigma$ is a conjugation preserving $\mathcal{G}$, it is clear that there is an isomorphism of $S$ with $\mathbb{C}^{2p}$, such that $\mathcal{G}^\sigma \simeq$ sp$(2p, \mathbb{R})$.
2. If $\sigma$ is a pseudoconjugation preserving $\mathcal{G}$, then $\mathcal{G}^\sigma \simeq$ usp$(p, q)$, $p + q = n = 2m$, $p = 2p'$, $q = 2q'$. All the real forms usp$(p, q)$ arise in this way. There is an isomorphism of $S$ with $\mathbb{C}^{2p}$ such that $\sigma$ goes to

$$(z_1, \ldots, z_m, z_{m+1}, \ldots, z_n) \mapsto J_m K_{p',q'}(z_1^*, \ldots z_m^*, z_{m+1}^*, \ldots, z_n^*),$$

where

$$J_m = \begin{pmatrix} 0 & I_{m \times m} \\ -I_{m \times m} & 0 \end{pmatrix}, \qquad K_{p',q'} = \begin{pmatrix} -I_{p' \times p'} & 0 & 0 & 0 \\ 0 & I_{q' \times q'} & 0 & 0 \\ 0 & 0 & -I_{p' \times p'} & 0 \\ 0 & 0 & 0 & I_{q' \times q'} \end{pmatrix}.$$

At the end of Section 2.1 we saw that there is a conjugation on $S$ when the spinors are real and a pseudoconjugation when they are quaternionic (both denoted by $\sigma$). We have a group, O$(n, \mathbb{C})$, Sp$(2p, \mathbb{C})$ or GL$(n, \mathbb{C})$ acting on $S$ and containing Spin$(V)^\mathbb{C}$. We note that this group is minimal in the classical group series. If the Lie algebra $\mathcal{G}$ of this group is stable under the conjugation

$$X \mapsto \sigma X \sigma^{-1}$$

then the real Lie algebra $\mathcal{G}^\sigma$ acts on $S^\sigma$ and contains the Lie algebra of Spin$(V)$. We shall call it the Spin$(V)$-algebra.

Let $B$ be the space of Spin$(V)^\mathbb{C}$-invariant bilinear forms on $S$. Since the representation on $S$ is irreducible, this space is at most one-dimensional. Let it be one-dimensional and let $\sigma$ be a conjugation or a pseudoconjugation and let $\psi \in B$. We define a conjugation in the space $B$ as

$$B \to B, \qquad \psi \mapsto \psi^\sigma, \quad \psi^\sigma(v, u) = \psi(\sigma(v), \sigma(u))^*.$$

It is then immediate that we can choose $\psi \in B$ such that $\psi^\sigma = \psi$. Then if $X$ belongs to the Lie algebra preserving $\psi$, so does $\sigma X \sigma^{-1}$.

### 6.2. Spin$(s, t)$-algebras

We now determine the real Lie algebras in each case. All the possible cases must be studied separately. We start with odd dimensions. All dimension and signature relations are

mod(8). In the following, a relation like $\mathrm{Spin}(V) \subseteq G$ for a group $G$ will mean that the image of $\mathrm{Spin}(V)$ under the spinor representation is in the connected component of $G$. The same applies for the relation $\mathrm{Spin}(V) \simeq G$.

### 6.2.1. Orthogonal spinors in odd dimension, $D_0 = 1, 7$

*Real spinors*, $\rho_0 = 1, 7$. There is a conjugation $\sigma$ on $S$ commuting with $\mathrm{Spin}(V)$. Then $\mathrm{Spin}(V) \subseteq \mathrm{SO}(S^\sigma) \simeq \mathrm{SO}(p, q)$. To determine $p$ and $q$, we look at the embedding of the maximal compact subgroup of $\mathrm{Spin}(V)$ into $\mathrm{SO}(p) \times \mathrm{SO}(q)$. We have three cases:

1. If $\rho = D$ ($s$ or $t$ is zero), $\mathrm{Spin}(V)$ is compact and it is embedded in the compact orthogonal group,

    $$\mathrm{Spin}(V) \subseteq \mathrm{SO}(2^{(D-1)/2}, \mathbb{R}),$$

    so $p$ or $q$ is zero. This is clear since the lowest dimensional spinor type representation of $\mathrm{Spin}(V)$ is $\mathbf{2}^{(D-1)/2}$.

2. If $s$ or $t$ is 1, then the maximal compact subgroup of $\mathrm{Spin}(V^D)$ is $\mathrm{Spin}(V^{D-1})$. Let $\varepsilon$ be the non-trivial central element of $\mathrm{Spin}(V^D)$ which maps to the identity under the homomorphism $\mathrm{Spin}(V^D) \to \mathrm{SO}(V^D)$. Under the injection

    $$\mathrm{Spin}(V^D) \to \mathrm{SO}(S^\sigma) \simeq \mathrm{SO}(p, q),$$

    the central element $\varepsilon$ maps to $-\mathcal{I}_{p+q}$. The compact subgroup of $\mathrm{Spin}(V^D)$ maps into the maximal compact subgroup of $\mathrm{SO}(p, q)$, so that

    $$\mathrm{Spin}(V^{D-1}) \to \mathrm{SO}(p) \times \mathrm{SO}(q).$$

    But the dimension of any spinor type representation of $\mathrm{Spin}(V^{D-1})$ is bigger or equal than $2^{(D-1)/2-1}$. Since $\varepsilon$ maps to $-\mathcal{I}_p \oplus -\mathcal{I}_q$, both maps

    $$\mathrm{Spin}(V^{D-1}) \to \mathrm{SO}(p) \quad \text{and} \quad \mathrm{Spin}(V^{D-1}) \to \mathrm{SO}(q)$$

    are spinor type representations. It follows that $p, q \geq 2^{(D-1)/2-1}$, so $p = q = 2^{(D-1)/2-1}$. So

    $$\mathrm{Spin}(V^D) \subseteq \mathrm{SO}(2^{(D-1)/2-1}, 2^{(D-1)/2-1}).$$

3. If $s, t \geq 2$, the maximal compact subgroup of $\mathrm{Spin}(V^D)$ is $\mathrm{Spin}(s) \times \mathrm{Spin}(t)/(\varepsilon_s = \varepsilon_t)$, where $\varepsilon_s$ and $\varepsilon_t$ are the central elements in $\mathrm{Spin}(s)$ and $\mathrm{Spin}(t)$, respectively, and they must be identified with $\varepsilon$. The embedding of the maximal compact subgroup must be

    $$\frac{\mathrm{Spin}(s) \times \mathrm{Spin}(t)}{\varepsilon_s = \varepsilon_t} \to \mathrm{SO}(p) \times \mathrm{SO}(q).$$

    The spinor type representation of $\mathrm{Spin}(s) \times \mathrm{Spin}(t)/(\varepsilon_s = \varepsilon_t)$ of minimal dimension is $\mathbf{2}^{(s-1)/2} \otimes \mathbf{2}^{t/2-1}$ if $s$ is odd and $t$ even (only with a tensor product representation is possible to identify $\varepsilon_s$ and $\varepsilon_t$). For the same reason that in (2), we have that $p = q = 2^{(D-1)/2-1}$. So

    $$\mathrm{Spin}(V) \subseteq \mathrm{SO}(2^{(D-1)/2-1}, 2^{(D-1)/2-1}).$$

Low dimensional examples are

$$\text{Spin}(4, 3) \subset \text{SO}(4, 4), \qquad \text{Spin}(8, 1) \subset \text{SO}(8, 8).$$

We give now a more complicated example. Consider the group $\text{Spin}(12, 5)$. The spinor representation is **256**, and should be embedded in the vector representation of $\text{SO}(p, q)$, $p + q = 256$. We have the following decomposition

$$\mathbf{256} \underset{\text{Spin}(12) \times \text{Spin}(5)}{\rightarrow} (\mathbf{32^+}, \mathbf{4}) \oplus (\mathbf{32^-}, \mathbf{4}).$$

It follows that $p = q = 128$, so the group will be $\text{SO}(128, 128)$.

Note that the representations of $\text{Spin}(12)$ and $\text{Spin}(5)$ are quaternionic separately, but when tensoring them a reality condition can be imposed.

Since there is no symmetric morphism $S \otimes S \to \Lambda^2$ one cannot construct in this case a simple superalgebra containing the orthogonal group.

*Quaternionic spinors*, $\rho_0 = 3, 5$. We have that $\text{Spin}(V)$ commutes with a pseudoconjugation on $S$. It then follows that

$$\text{Spin}(V) \subseteq \text{SO}^*(2^{(D-1)/2}).$$

A low dimensional example is

$$\text{Spin}(6, 1) \subset \text{SO}^*(8), \qquad \text{Spin}(5, 2) \subset \text{SO}^*(8).$$

We explicitly compute another example, the group $\text{Spin}(10, 5)$ whose quaternionic spinor representation is **128**. We have the following decomposition:

$$\mathbf{128} \underset{\text{Spin}(10) \times \text{Spin}(5)}{\rightarrow} (\mathbf{16^+}, \mathbf{4}) \oplus (\mathbf{16^-}, \mathbf{4}).$$

### 6.2.2. Symplectic spinors in odd dimension, $D_0 = 3, 5$

*Real spinors*, $\rho_0 = 1, 7$. Since there is a conjugation commuting with $\text{Spin}(V)$,

$$\text{Spin}(V) \subseteq \text{Sp}(2^{(D-1)/2}, \mathbb{R}).$$

We have the low dimensional examples

$$\text{Spin}(2, 1) \simeq \text{SL}(2, \mathbb{R}), \qquad \text{Spin}(3, 2) \simeq \text{Sp}(4, \mathbb{R}).$$

*Quaternionic spinors*, $\rho_0 = 3, 5$. $\text{Spin}(V)$ commutes with a pseudoconjugation, so $\text{Spin}(V) \subseteq \text{USp}(p, q)$. We have three cases,

1. If $s$ or $t$ is zero, then $\text{Spin}(V)$ is compact and

$$\text{Spin}(V) \subseteq \text{USp}(2^{(D-1)/2}).$$

Low dimensional examples are

$$\text{Spin}(3) \simeq \text{SU}(2), \qquad \text{Spin}(5) \simeq \text{USp}(4).$$

2. If $s$ or $t$ are 1, then the maximal compact subgroup is $\mathrm{Spin}(V^{D-1})$ is embedded in $\mathrm{USp}(p) \times \mathrm{USp}(q)$. The same reasoning as in the orthogonal case can be applied here, and $p = q = 2^{(D-1)/2-1}$. So

$$\mathrm{Spin}(V) \subseteq \mathrm{USp}(2^{(D-1)/2-1}, 2^{(D-1)/2-1}).$$

Low dimensional examples are

$$\mathrm{Spin}(4, 1) \simeq \mathrm{USp}(2, 2).$$

3. If $s, t \geq 2$, then

$$\frac{\mathrm{Spin}(s) \times \mathrm{Spin}(t)}{\varepsilon_s = \varepsilon_t} \to \mathrm{USp}(p) \times \mathrm{USp}(q).$$

As before, $p = q = 2^{(D-1)/2-1}$ and so

$$\mathrm{Spin}(V) \subseteq \mathrm{USp}(2^{(D-1)/2-1}, 2^{(D-1)/2-1}).$$

We analyze now the even dimensional cases.

### 6.2.3. Orthogonal spinors in even dimensions, $D_0 = 0$

*Real spinors*, $\rho_0 = 0$. The group $\mathrm{Spin}(V)^{\pm}$ (projections of $\mathrm{Spin}(V)$ with the chiral or Weyl representations) commutes with a conjugation. Using the same reasoning as in the odd case, we have that for a compact group

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{SO}(2^{D/2-1}).$$

An example of this is $\mathrm{Spin}(8) \simeq \mathrm{SO}(8)$.[2] For a non-compact group we have

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{SO}(2^{D/2-2}, 2^{D/2-2}),$$

as for example $\mathrm{Spin}(4, 4) \simeq \mathrm{SO}(4, 4)$.

*Quaternionic spinors*, $\rho_0 = 4$. We note that $s$ and $t$ are both even and that neither can be zero. $\mathrm{Spin}(V)^{\pm}$ commutes with a pseudoconjugation, so

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{SO}^*(2^{D/2-1}).$$

An example is $\mathrm{Spin}(6, 2) \simeq \mathrm{SO}^*(8)$.

*Complex spinors*, $\rho_0 = 2, 6$. $s$ and $t$ must be bigger than zero. $\mathrm{Spin}(V)^{\pm}$ does not commute with a conjugation or pseudoconjugation, since it is not real nor quaternionic. It follows that there is no real form of $\mathrm{SO}(2^{D/2-1}, \mathbb{C})$ containing $\mathrm{Spin}(V)^{\pm}$. We have instead, using [28]

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{SO}(2^{D/2-1}, \mathbb{C})_{\mathbb{R}},$$

which is also a simple real group. (The suffix "$\mathbb{R}$" means that the complex group is considered as a real Lie group.) It cannot be seen as a real form of any complex simple Lie group [29]. As an example,

$$\mathrm{Spin}(7, 1) \subset \mathrm{SO}(8, \mathbb{C})_{\mathbb{R}}.$$

---

[2] Notice that for $D = 8$ one has the phenomenon of triality.

### 6.2.4. Symplectic spinors in even dimensions, $D_0 = 4$

*Real spinors*, $\rho_0 = 0$. $p$ and $q$ are both even and neither can be zero. We have

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{Sp}(2^{D/2-1}, \mathbb{R}).$$

The lowest dimensional case is not simple,

$$\mathrm{Spin}(2,2) \simeq \mathrm{Sp}(2, \mathbb{R}) \times \mathrm{Sp}(2, \mathbb{R}).$$

*Quaternionic spinors*, $\rho_0 = 4$. $\mathrm{Spin}(V)^{\pm}$ commute with a pseudoconjugation, so $\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{USp}(p, q)$, $p + q = 2^{D/2-1}$. Again, the lowest dimensional case is semisimple,

$$\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2).$$

If $s$ or $t$ are zero we are in the compact case and

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{USp}(2^{D/2-1}).$$

The other possible case is $s, t > 0$, and then $s, t \geq 4$. As in the even case, we have $p = q = 2^{D/2-2}$,

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{USp}(2^{D/2-2}, 2^{D/2-2}).$$

*Complex spinors*, $\rho_0 = 2, 6$. As in the orthogonal case, no real form of $\mathrm{Sp}(2^{D/2-1}, \mathbb{C})$ containing $\mathrm{Spin}(V)^{\pm}$ exists. We have the embedding

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{Sp}(2^{D/2-1}, \mathbb{C})_{\mathbb{R}}.$$

An example is

$$\mathrm{Spin}(3,1) = \mathrm{Sp}(2, \mathbb{C})_{\mathbb{R}} \simeq \mathrm{SL}(2, \mathbb{C})_{\mathbb{R}}.$$

### 6.2.5. Linear spinors, $D_0 = 2, 6$

*Real spinors*, $\rho_0 = 0$. $\mathrm{Spin}(V)^{\pm}$ commutes with a conjugation, so one has an embedding into the standard real form of the linear group,

$$\mathrm{Spin}(V)^{\pm} \subset \mathrm{SL}(2^{D/2-1}, \mathbb{R}).$$

As an example, we have $\mathrm{Spin}(3,3)^{\pm} \simeq \mathrm{SL}(4, \mathbb{R})$.

*Quaternionic spinors*, $\rho_0 = 4$. The representations $S^{\pm}$ are dual to each other and they commute with a pseudoconjugation. They leave no bilinear form invariant. If there were an invariant Hermitian form $\langle \cdot, \cdot \rangle$ then one could define an invariant bilinear form:

$$(s_1, s_2) = \langle \sigma s_1, s_2 \rangle.$$

So the only possibility is

$$\mathrm{Spin}(V)^{\pm} \subseteq \mathrm{SU}^*(2^{D/2-1}).$$

A low dimension example is $\mathrm{Spin}(5,1) \simeq \mathrm{SU}^*(4)$.

*Complex spinors*, $\rho_0 = 2, 6$. We will denote by $(\cdot, \cdot)$ the $\mathrm{Spin}(V)$-invariant pairing between $S^-$ and $S^+$. We remind that on $S = S^+ \oplus S^-$ there is a conjugation $\sigma$ commuting

with the action of Spin$(V)$ (see Section 2). It satisfies $\sigma(S^{\pm}) = S^{\mp}$, so we can define a Spin$(V)^{\pm}$-invariant sesquilinear form on $S^+$,

$$\langle s_1^+, s_2^+ \rangle = (\sigma(s_1^+), s_2^+), \quad s_i^+ \in S^+.$$

By irreducibility of the action of Spin$(V)^{\pm}$, the space of invariant sesquilinear forms is one-dimensional. We can choose an Hermitian form as a basis, so it follows that

$$\text{Spin}(V)^{\pm} \subseteq \text{SU}(p, q).$$

If $s$ or $t$ are zero (compact case), then we have that $p, q \geq 2^{D/2-1}$, so either $p$ or $q$ are zero and

$$\text{Spin}(V)^{\pm} \subseteq \text{SU}(2^{D/2-1}).$$

If neither $p$ nor $q$ is zero, then $p, q \geq 2$ and even. We have that the embedding of the maximal compact subgroup must be

$$\frac{\text{Spin}(p) \times \text{Spin}(q)}{\varepsilon_p = \varepsilon_q} \subseteq \text{S}(\text{U}(p) \times \text{U}(q)).$$

So $p, q \geq 2^{p/2-1} \times 2^{q/2-1} = 2^{D/2-2}$. It follows that

$$\text{Spin}(V)^{\pm} \subseteq \text{SU}(2^{D/2-2}, 2^{D/2-2}).$$

We have the low dimensional examples

$$\text{Spin}(6) \simeq \text{SU}(4), \qquad \text{Spin}(4, 2) \simeq \text{SU}(2, 2).$$

## 7. Spin(V) superalgebras

We now consider the embedding of Spin$(V)$ in simple real superalgebras. We require in general that the odd generators are in a real spinor representation of Spin$(V)$. In the cases $D_0 = 2, 6$, $\rho_0 = 0, 4$ we have to allow the two independent irreducible representations, $S^+$ and $S^-$ in the superalgebra, since the relevant morphism is

$$S^+ \otimes S^- \to \Lambda^2.$$

The algebra is then non-chiral.

We first consider minimal superalgebras, i.e. those with the minimal even subalgebra. From the classification of simple superalgebras [30,31] one obtains the results listed in Table 7.

We note that the even part of the minimal superalgebra contains the Spin$(V)$ algebra obtained in Section 6.2 as a simple factor. For all quaternionic cases, $\rho_0 = 3, 4, 5$, a second simple factor SU(2) or SO*(2) is present. For the linear cases there is an additional non-simple factor, SO(1, 1) or U(1), as discussed in Section 6.2.

For $D = 7$ and $\rho = 3$ there is actually a smaller superalgebra, the exceptional superalgebra $f(4)$ with bosonic part spin$(5, 2) \times$ su(2). The superalgebra appearing in Table 7 belongs to the classical series and its even part is so*(8) $\times$ su(2), being so*(8) the Spin(5, 2)-algebra.

Table 7
Minimal Spin(V) superalgebras

| $D_0$ | $\rho_0$ | Spin(V) algebra | Spin(V) superalgebra |
|---|---|---|---|
| 1, 7 | 1, 7 | $so(2^{(D-3)/2}, 2^{(D-3)/2})$ | |
| 1, 7 | 3, 5 | $so^*(2^{(D-1)/2})$ | $osp(2^{(D-1)/2})^*|2)$ |
| 3, 5 | 1, 7 | $sp(2^{(D-1)/2}, \mathbb{R})$ | $osp(1|2^{(D-1)/2}, \mathbb{R})$ |
| 3, 5 | 3, 5 | $usp(2^{(D-3)/2}, 2^{(D-3)/2})$ | $osp(2^*|2^{(D-3)/2}, 2^{(D-3)/2}$ |
| 0 | 0 | $so(2^{(D-4)/2}, 2^{(D-4)/2})$ | |
| 0 | 2, 6 | $so(2^{(D-2)/2}, \mathbb{C})^{\mathbb{R}}$ | |
| 0 | 4 | $so^*(2^{(D-2)/2})$ | $osp(2^{(D-2)/2})^*|2)$ |
| 2, 6 | 0 | $sl(2^{(D-2)/2}, \mathbb{R})$ | $sl(2^{(D-2)/2}|1)$ |
| 2, 6 | 2, 6 | $su(2^{(D-4)/2}, 2^{(D-4)/2})$ | $su(2^{(D-4)/2}, 2^{(D-4)/2}|1)$ |
| 2, 6 | 4 | $su^*(2^{(D-2)/2}))$ | $su(2^{(D-2)/2})^*|2)$ |
| 4 | 0 | $sp(2^{(D-2)/2}, \mathbb{R})$ | $osp(1|2^{(D-2)/2}, \mathbb{R})$ |
| 4 | 2, 6 | $sp(2^{(D-2)/2}, \mathbb{C})^{\mathbb{R}}$ | $osp(1|2^{(D-2)/2}, \mathbb{C})$ |
| 4 | 4 | $usp(2^{(D-4)/2}, 2^{(D-4)/2})$ | $osp(2^*|2^{(D-4)/2}, 2^{(D-4)/2}$ |

Since we are considering minimal simple superalgebras, there are some terms in the anticommutator that in principle are allowed morphisms but that do not appear. One can see that these are

$$D_0 = 2, 6, \quad \rho_0 = 0, 2, 4, 6, \quad S^{\pm} \otimes S^{\pm} \to \sum_k \Lambda^{2k+1},$$

$$D_0 = 4, \quad \rho_0 = 0, 2, 6, \quad S^+ \otimes S^- \to \sum_k \Lambda^{2k+1},$$

$$D_0 = 1, 7, \quad \rho_0 = 3, 5, \quad S \otimes S \to \sum_{k \neq 0} \Lambda^{4k},$$

$$D_0 = 0, \quad \rho_0 = 4, \quad S^+ \otimes S^+ \to \sum_{k \neq 0} \Lambda^{4k}.$$

Keeping the same number of odd generators, the maximal simple superalgebra containing Spin(V) is an orthosymplectic algebra with Spin(V) $\subset$ Sp(2n, $\mathbb{R}$), being 2n the real dimension of S. The various cases are displayed in Table 8. We note that the minimal superalgebra is not a subalgebra of the maximal one, although it is so for the bosonic parts.

Table 8
Maximal Spin(V) superalgebras

| $D_0$ | $\rho_0$ | Orthosymplectic |
|---|---|---|
| 3, 5, | 1, 7 | $osp(1|2^{(D-1)/2}, \mathbb{R})$ |
| 1, 7 | 3, 5 | $osp(1|2^{(D+1)/2}, \mathbb{R})$ |
| 0 | 4 | $osp(1|2^{D/2}, \mathbb{R})$ |
| 4 | 0 | $osp(1|2^{(D-2)/2}, \mathbb{R})$ |
| 4 | 2, 6 | $osp(1|2^{D/2}, \mathbb{R})$ |
| 2, 6 | 0 | $osp(1|2^{D/2}, \mathbb{R})$ |
| 2, 6 | 4 | $osp(1|2^{(D+2)/2}, \mathbb{R})$ |
| 2, 6 | 2, 6 | $osp(1|2^{D/2}, \mathbb{R})$ |

Tables 7 and 8 show that there are 12 (mod(8) in $D$ and $\rho$) superalgebras for $D$ even and 8 mod(8) superalgebras for $D$ odd, in correspondence with Table 5.

## 8. Summary

In this paper we have considered superalgebras containing space–time supersymmetry in arbitrary dimensions and with arbitrary signature. In particular $D$-dimensional super conformal algebras give, by Inonü–Wigner contraction, super translation algebras with central charges in $(D + 1)$ dimensions. They also contain $D$-dimensional super Poincaré algebras as subalgebras. The maximal central extension of the Poincaré superalgebra can be obtained by contraction of the osp$(1|2n, \mathbb{R})$ superalgebra where $n$ is related to the space dimensions according to (8).

In Table 9 we report these superalgebras for a physical space–time of signature $(D-1, 1)$, $D = 3, \ldots, 12$. The first column (Lorentz) is the supersymmetric extension of the orthogonal algebra so$(D-1, 1)$. The second column (Conformal) is the supersymmetric extension of the conformal algebra in dimension $D$, so$(D, 2)$. The third column (Orthosymplectic) is the superalgebra that by contraction gives the maximal central extension of the super translation algebra in dimension $D$. Note that the same algebras appear in $D = 3, 11$ and in $D = 4, 12$, owing to the mod(8) periodicity.

Note that the Poincaré supersymmetries obtained by contractions of the orthosymplectic algebras in Table 9 are non-chiral for $D = 6, 10$ and are $N = 2$ for $D = 8, 9$. We can compare Table 9 with Table 7 of Ref. [3] dealing with $D = 10, 11, 12$. We find general agreement although in Ref. [3] the real forms of the supergroups were not worked out. Furthermore in the case of Lorentz superalgebra in $D = 12$ our analysis shows that the result is osp$(1|32, \mathbb{C})$. Note that in $D = 4$ we get both, the Wess–Zumino $N = 1$ super conformal algebra [36] and by contraction of osp$(1|4, \mathbb{R})$, the Poincaré superalgebra with the domain wall central charge [37,38].

It is worthwhile to mention that from our tables we can retrieve the super conformal algebras that do not violate the Coleman–Mandula [32] theorem and its supersymmetric version, the Haag–Lopuszańsky–Sohnius theorem [33]. These state that the even part of

Table 9
Supersymmetric extensions of space–time groups

| $D$ | Lorentz | Conformal | Orthosymplectic |
|---|---|---|---|
| 3 | osp$(1|2, \mathbb{R})$ | osp$(1|4, \mathbb{R})$ | osp$(1|2, \mathbb{R})$ |
| 4 | osp$(1|2, \mathbb{C})$ | su$(2, 2|1)$ | osp$(1|4, \mathbb{R})$ |
| 5 | | osp$(8^*|2)$ | osp$(1|8, \mathbb{R})$ |
| 6 | su$(4^*|2)$ | osp$(8^*|2)$ | osp$(1|16, \mathbb{R})$ |
| 7 | osp$(8^*|2)$ | osp$(16^*|2)$ | osp$(1|16, \mathbb{R})$ |
| 8 | | su$(8, 8|1)$ | osp$(1|32, \mathbb{R})$ |
| 9 | | osp$(1|32, \mathbb{R})$ | osp$(1|32, \mathbb{R})$ |
| 10 | sl$(16|1)$ | osp$(1|32, \mathbb{R})$ | osp$(1|32, \mathbb{R})$ |
| 11 | osp$(1|32, \mathbb{R})$ | osp$(1|64, \mathbb{R})$ | osp$(1|32, \mathbb{R})$ |
| 12 | osp$(1|32, \mathbb{C})$ | su$(32, 32|1)$ | osp$(1|64, \mathbb{R})$ |

the superalgebra should be given by the sum of the space–time symmetry algebra and an internal symmetry algebra. It is immediately seen that this happens only for $D = 3, 4, 6$. Indeed, this occurs because of the following isomorphisms:

$$SO(3, 2) \simeq Sp(4, \mathbb{R}), \qquad SO(4, 2) \simeq SU(2, 2), \qquad SO(6, 2) \simeq SO^*(8).$$

The $D = 5$ case is also allowed if we replace the $osp(8^*|2)$ superalgebra with the exceptional superalgebra $f(4)$. The first departure occurs at $D = 7$, where the conformal group $SO(7, 2)$ must be embedded in $SO^*(16)$ to find a supersymmetric extension.

## Acknowledgements

## Appendix A

Some embeddings of real forms of non-compact groups which have been used through the text are given below.

$$SO(p + q, \mathbb{C})_{\mathbb{R}} \supset SO(p, q), \quad SO(2n, \mathbb{C})_{\mathbb{R}} \supset SO^*(2n), \quad SO(n, n) \supset SO(n, \mathbb{C})_{\mathbb{R}},$$
$$SO(n, n) \supset SL(n, \mathbb{R}) \times SO(1, 1) = GL(n, \mathbb{R}), \quad SO(4n, 4n) \supset SU(2) \times Usp(2n, 2n),$$
$$SO^*(2p + 2q) \supset SU(p, q) \times U(1), \qquad SO^*(2n) \supset SO(n, \mathbb{C})_{\mathbb{R}},$$
$$SO^*(4n) \supset SU^*(2n) \times SO(1, 1), \qquad Sp(2p + 2q, \mathbb{C})_{\mathbb{R}} \supset Usp(2p, 2q),$$
$$Sp(2p + 2q, \mathbb{R}) \supset U(p, q), \quad Sp(2n, \mathbb{R}) \supset GL(n, \mathbb{R}), \quad Sp(4n, \mathbb{R}) \supset Sp(2n, \mathbb{C})_{\mathbb{R}},$$
$$Sp(4n, \mathbb{R}) \supset SU(2) \times SO^*(2n), \qquad Usp(2n, 2n) \supset SU^*(2n) \times SO(1, 1).$$

## Appendix B

We give explicitly the decomposition of the tensor product representation $S \otimes S$. The Clifford algebra has a $\mathbb{Z}_2$ grading,

$$\mathcal{C}(s, t) = \mathcal{C}^+(s, t) \oplus \mathcal{C}^-(s, t),$$

where

$$\mathcal{C}^+(s, t) = \sum_k \Lambda^{2k}, \qquad \mathcal{C}^-(s, t) = \sum_k \Lambda^{2k+1}.$$

Table 10
Symmetries of gamma matrices for $D$ even

|  | $D_0 = 0$ | $D_0 = 2$ | $D_0 = 4$ | $D_0 = 6$ |
|---|---|---|---|---|
| $\mathcal{A}_0$ | $+$ | $\pm$ | $-$ | $\mp$ |
| $\mathcal{A}_1$ | $\pm$ | $+$ | $\mp$ | $-$ |
| $\mathcal{A}_2$ | $-$ | $\mp$ | $+$ | $\pm$ |
| $\mathcal{A}_3$ | $\mp$ | $-$ | $\pm$ | $+$ |

*D odd.* $\mathcal{C}^\pm$ carry isomorphic representations of Spin$(s, t)$, since $\Lambda^k \approx \Lambda^{D-k}$. We can consider only $\mathcal{C}^+$. We have then

$$\mathcal{C}^+ = \mathcal{A}_0 + \mathcal{A}_2, \quad \mathcal{A}_0 = \sum_k \Lambda^{4k}, \quad \mathcal{A}_2 = \sum_k \Lambda^{4k+2}.$$

Form Table 3 it follows that the morphisms in $\mathcal{A}_0$ are symmetric for $D_0 = 1, 7$ and anti-symmetric for $D_0 = 3, 5$. $\mathcal{A}_2$ is symmetric for $D_0 = 3, 5$ and antisymmetric for $D_0 = 1, 7$.

*D even.* $\mathcal{C}^+$ is not isomorphic to $\mathcal{C}^-$.

$$\mathcal{C}^+ = \mathcal{A}_0 + \mathcal{A}_2, \quad \mathcal{A}_0 = \sum_k \Lambda^{4k}, \quad \mathcal{A}_2 = \sum_k \Lambda^{4k+2},$$

$$\mathcal{C}^- = \mathcal{A}_1 + \mathcal{A}_3, \quad \mathcal{A}_1 = \sum_k \Lambda^{4k+1}, \quad \mathcal{A}_3 = \sum_k \Lambda^{4k+3}.$$

The symmetry properties are given in Table 10.

$+$ means symmetric and $-$ antisymmetric. $\pm$ and $\mp$ are symmetric or antisymmetric, respectively, depending on the choice of the charge conjugation matrix (see Section 2.2).

The morphisms are

$$S^\pm \otimes S^\pm \to \mathcal{C}^+, \quad D = 0, 4, \qquad S^\pm \otimes S^\pm \to \mathcal{C}^-, \quad D = 2, 6,$$

$$S^\pm \otimes S^\mp \to \mathcal{C}^+, \quad D = 2, 6, \qquad S^\pm \otimes S^\mp \to \mathcal{C}^-, \quad D = 0, 4$$

The Spin$(V)$-algebra is the module $\mathcal{A}_2$. The compact generators for the case of Minkowskian signature $(D-1, 1)$, are given by the space like components of the even generators, $Z_{[i_1 \cdots i_k]}$, $i_j = 1, \ldots, D - 1$. The maximal compact subgroups are

$D = 3$   U(1)
$D = 4$   SU(2)
$D = 5$   SU(2) × SU(2)
$D = 6$   USp(4)
$D = 7$   U(4)
$D = 8$   SO(8)
$D = 9$   SO(8) × SO(8)
$D = 10$   SO(16)
$D = 11$   U(16)
$D = 12$   USp(32).

This is in agreement with Table 7.

# References

[1] P. Horava, E. Witten, Heterotic and type I string dynamics from eleven dimensions, Nucl. Phys. B 460 (1996) 506.

[2] R. D'Auria, P. Frè, Geometric supergravity in $D = 11$ and its hidden supergroup, Nucl. Phys. B 201 (1982) 101.

[3] J.W. Van Holten, A. Van Proeyen, $N = 1$ supersymmetry algebras in $D = 2$, $D = 3$, $D = 4$ mod(8), J. Phys. A 15 (1982) 3763.

[4] R. Finkelstein, M. Villasante, Majorana spinors in higher dimensional theories, Phys. Rev. D 31 (1985) 425.

[5] A. Van Proeyen, Tools for Supersymmetry. hep-th/9910030.

[6] L. Andrianopoli, M. Derix, G.W. Gibbons, C. Herdeiro, A. Santambrogio, A. Van Proeyen, Isometric embedding of BPS branes in flat spaces with two times, Class. Quant. Grav. 17 (2000) 1875.

[7] P.K. Townsend, M(embrane) theory on $T^9$, Nucl. Phys. Proc. Suppl. 68 (1998) 11.

[8] P.K. Townsend, in: Proceedings of the March 95 PASCOS/Johns Hopkins Conference on p-Brane Democracy. hep-th/9507048.

[9] C. Vafa, Evidence for F-theory, Nucl. Phys. B 469 (1996) 403.

[10] I. Bars, Two-time physics in field theory, Phys. Rev. D 62 (2000) 046007; A case for 14 dimensions, Phys. Lett. B 403 (1997) 257–264; S-theory, Phys. Rev. D 55 (1997) 2373–2381.

[11] I. Bars, M. Günaydin, Construction of Lie algebras and Lie superalgebras from ternary algebras, J. Math. Phys. 20 (9) (1979) 1977.

[12] M.P. Blencowe, M.J. Duff, Supermembranes and the signature of space–time, Nucl. Phys. B 310 (1988) 387.

[13] T. Regge, in: Proceedings of Les Houches on Relativity, Groups and Topology, 1984, p. 933.

[14] P. Van Nieuwenhuizen, in: Proceedings of Les Houches on Relativity, Groups and Topology, 1984, p. 823.

[15] C.M. Hull, Massive string theories from M-theory and F-theory, JHEP 9811027 (1998).

[16] P. Horava, M-theory as a holographic field theory, Phys. Rev. D 59 (1999) 046004.

[17] M. Günaydin, Unitary supermultiplets of OSp(1/32,R) and M-theory, Nucl. Phys. B 528 (1998) 432.

[18] S. Ferrara, M. Porrati, AdS(5) superalgebras with brane charges, Phys. Lett. B 458 (1999) 43.

[19] S. Ferrara, M. Porrati, Central extensions of supersymmetry in four and three dimensions, Phys. Lett. B 423 (1998) 255.

[20] E. Bergshoeff, A. Van Proeyen, The many faces of OSp(1–32), Class. Quant. Grav. 17 (2000) 3277.

[21] P. West, Hidden superconformal symmetry in M theory, JHEP 0008007 (2000).

[22] W. Nahm, Supersymmetries and their representations, Nucl. Phys. B 135 (1978) 149.

[23] J. Strathdee, Extended Poincare supersymmetry, Int. J. Mod. Phys. A 2 (1) (1987) 273.

[24] C. Chevalley, The Algebraic Theory of Spinors, Columbia University Press, New York, 1954.

[25] N. Bourbaki, Algebre, Hermann, Paris, 1958 (Chapter 9).

[26] R. Coquereaux, Modulo 8 periodicity of real Clifford algebras and particle physics, Phys. Lett. B 115 (1982) 389.

[27] Y. Choquet-Bruhat, C. DeWitt-Morette, Analysis, Manifolds and Physics. Part II. Applications, North-Holland, Amsterdam, 1989.

[28] P. Deligne, Notes on spinors, in: Quantum Fields and Strings: A Course for Mathematicians, American Mathematical Society, Providence, RI, 1999.

[29] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York, 1978.

[30] W. Nahm, V. Rittenberg, M. Scheunert, Classification of all simple graded Lie algebras whose Lie algebra is reductive, J. Math. Phys. 17 (1976) 1626.

[31] V.G. Kac, A Sketch of Lie superalgebra theory, Commun. Math. Phys. 53 (1977) 31; Lie superalgebras, Adv. Math. 26 (1977) 8; J. Math. Phys. 21 (1980) 689.

[32] S. Coleman, J. Mandula, All possible symmetries of the $S$ matrix, Phys. Rev. 159 (1967) 1251.

[33] R. Haag, J. Lopuszański, M. Sohnius, All possible generators of supersymmetries of the $S$ matrix, Nucl. Phys. B 88 (1975) 257.

[34] B. Craps, J. Gomis, D. Mateos, A. Van Proeyen, BPS solutions of a D5-brane world volume in a D3-brane background from superalgebras, JHEP 9904004 (1999).

[35] J.A. de Azcárraga, J.P. Gauntlett, J.M. Izquierdo, P.K. Townsend, Topological extensions of the supersymmetry algebra for extended objects, Phys. Rev. Lett. 63 (1989) 2443.

[36] J. Wess, B. Zumino, Supergauge transformations in four-dimensions, Nucl. Phys. B 70 (1974) 39.

[37] G. Dvali, M. Shifman, Dynamical compactification as a mechanism of spontaneous supersymmetry breaking, Nucl. Phys. B 504 (1997) 127.

[38] G. Dvali, M. Shifman, Domain walls in strongly coupled theories, Phys. Lett. B 396 (1997) 64 .

[39] V. Cortés, D. Alekseevsky, Classification of $N$-(super)-extended poincaré algebras and bilinear invariants of the spinor representation of spin $(p, q)$, Commun. Math. Phys. 183 (1997) 477–510.